尺規作圖：從三角形到正方形！

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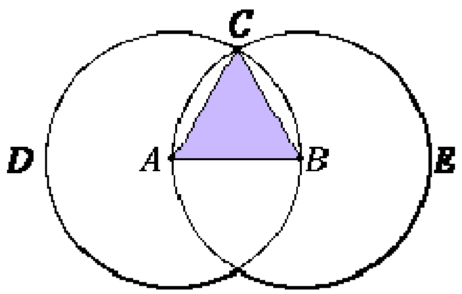
凡是在國中時期上過幾何課的人，應該都知道如何利用尺規作圖（或幾何作圖 (geometric construction)），在已知（或給定）線段上，求作一個正三角形（或等邊三角形）。依此類推，顯然，我們也應該很容易作一個正方形才是！我蠻好奇的，這一問題有多少人曾經認真想過？

事實上，非常令人驚奇，這一問題一點都不簡單！至少放在歐幾里得《幾何原本》的脈絡就是如此！

讓我們先審察《幾何原本》(*The Elements*) 中，歐幾里得 (Euclid) 如何作一個正三角形或等邊三角形）：

Prop. I. 1: ***To construct an equilateral triangle on a given finite straight line.***



Let *AB* be the given finite straight line.

It is required to construct an equilateral triangle on the straight line *AB.* Describe the circle *BCD* with center *A* and radius *AB.* Again describe the circle *ACE* with center *B* and radius *BA.* Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B.*

Describe the circle *BCD* with center *A* and radius *AB.* Again describe the circle *ACE* with center *B* and radius *BA.* Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B.* **(Post. 3, Post. 1)**

Now, since the point *A* is the center of the circle *CDB,* therefore *AC* equals *AB.* Again, since the point *B* is the center of the circle *CAE,* therefore *BC* equals *BA.* **(I. Def. 15)**

But *AC* was proved equal to *AB,* therefore each of the straight lines *AC* and *BC* equals *AB.*

And things which equal the same thing also equal one another, therefore *AC* also equals *BC.* **(C.N. 1)**

Therefore the three straight lines *AC, AB,* and *BC* equal one another.

Therefore the triangle *ABC* is equilateral, and it has been constructed on the given finite straight line *AB.* **(I. Def. 20)** **Q.E.F**

在上述引文（出自 Heath 版）中，Prop. I.1 指第 I 冊第一個命題（Prop.是 proposition 的縮寫），因此，正三角形的尺規作圖，乃是《幾何原本》全書的第一個命題，這對希臘數學的意義當然非同小可。原來，這是歐幾里得認爲可以被研究的幾何物件 (geometric entities)，莫過於那些可以經由尺規作出來的圖形。同時，由於它是全書第一個命題，所以，按照演繹科學 (deductive science，亞里斯多德的概念) 的規範，它必須只能運用定義 (definition)、設準 (postulate) 以及公理 (common notion，或譯爲共有概念) 來證明。而這些，正是《幾何原本》第 I 冊開頭的安排：依序爲 23 個定義，5 個設準，以及 5 個公理。因此，在上述引文中，Post. 1，Post. 3 分別代表設準 1 和 3，C. N. 1 代表公理 1，I. Def. 15，I. Def. 20 分別代表第 I 冊定義 15 和 20。至於 Q.E.F 則是拉丁字 quod erat faciendum 的縮寫，表示「得其所作」，亦即作圖完成的意思。

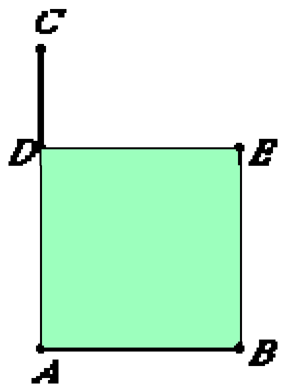
還有，我們也必須特別注意，歐幾里得的作圖題正如證明題一樣，不但都稱爲命題，而且都必須經過嚴密的證明。這或許可以解釋：何以正方形的作圖題必須安排在幾何原本第 I 冊的命題 46。請參看下面引文：

Prop. I. 46. ***To describe a square on a given straight line.***

Let *AB* be the given straight line.

It is required to describe a square on the straight line *AB.*

Draw *AC* at right angles to the straight line *AB* from the point *A* on it. Make *AD* equal to *AB.* Draw *DE* through the point *D* parallel to *AB,* and draw *BE* through the point *B* parallel to *AD.* **(I. 11, I. 3, I. 31)**



Then *ADEB* is a parallelogram. Therefore *AB* equals *DE,* and *AD* equals *BE.* **(I. 34)**

But *AB* equals *AD,* therefore the four straight lines *BA, AD, DE,* and *EB* equal one another. Therefore the parallelogram *ADEB* is equilateral.

I say next that it is also right-angled.

Since the straight line *AD* falls upon the parallels *AB* and *DE*, therefore the sum of the angles *BAD* and *ADE* equals two right angles. **(I. 29)**

But the angle *BAD* is right, therefore the angle *ADE* is also right.

And in parallelogrammic areas the opposite sides and angles equal one another, therefore each of the opposite angles *ABE* and *BED* is also right. Therefore *ADEB* is right-angled. (I. 34)

And it was also proved equilateral.

Therefore it is a square, and it is described on the straight line *AB.* **(I. Def. 22)**

Q.E.F

在上述證明中，如果我們將歐幾里得所運用的命題 I. 3，I. 11，I. 31 和 I. 34 之成立所必須依賴的前導命題，整體追溯彙整，則可以列出**求作一個正方形所必須假定的命題（不含設準和公理在內），總計有命題 1，2，3，4，5，7，8，10，11，13，15，16，18，20，22，23，26，27，29，31 和 34 等 21 個之多**。由此可見，從正三角形到正方形的尺規作圖，一點都不是那麼容易可以「依此類推」！這也說明：從論證幾何（譬如以《幾何原本》為範例）的觀點來看，直觀上有時相當「天真的」的「類推」(analogy)，並不見得永遠行得通！不過，這反過來更加證成 (justify) 幾何論證的重要意義！

這是歐幾里得留給我們的珍貴遺產，任何一位數學知識的愛好者都不應輕忽視之！

有關正 5、6、15 邊形之作圖，則出現在《幾何原本》第 IV 冊中，不過，這是下一篇文章的主題，我們下一次再談。

參考文獻

Heath, Thomas L. (1956). *Thirteen Books of Euclid's Elements*. New York: Dover Publications, INC.

Joyce, David: http://aleph0.clarku.edu/~djoyce.